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Semisimple Extensions and Elements of Trace 1

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INTRODUCTION

Let H be a Hopf algebra which is Frobenius over a commutative ring k. Let A be an H-module algebra, A^H its ring of invariants under the H-action, and $A \# H$ the associated semidirect, or smash, product. This setup has been shown by the authors and S. Montgomery [CF, CFM], to give rise to a Morita context $[A^H, A, A, A \# H]$ with structure maps $[\ ,\ ]: A \otimes_{A^H} A \to A \# H$, and $(\ ,\ ): A \otimes_{A \# H} A \to A^H$. The comodule theoretic concept of H-Galois extensions was shown [CFM] under these circumstance to be equivalent to surjectivity of $[\ ,\ ]$. In this paper we start out by showing that another important comodule theoretic concept, that of a total integral [D1], can be translated to a module theoretic concept as well. We show that existence of a total integral is equivalent to the surjectivity of $(\ ,\ )$, which we also termed suggestively: existence of an element $c \in A$ of trace 1. Furthermore, if this element c centralizes A^H, then A^H is an A^H-bimodule direct summand of A. We apply these to the Morita context associated with the H^*-module algebra $A \# H$, to prove the main theorem of this paper (Theorem 1.8). This theorem deals with semisimple extensions (sometimes coined "Maschke type" theorems), or better yet, with separable extensions. We show that when A/A^H is H^*-Galois,

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separability of the extension A/A^H is equivalent to the existence of a trace 1 element $\omega \in A \# H$ which centralizes A. This, in turn, is equivalent to A being an A-bimodule direct summand of $A \# H$. Any of these equivalent conditions imply that A/A^H is a semisimple extension. As a corollary we deduce a result of Doi: if A/A^H is H^*-Galois and $\varepsilon(T)=1$, $T \in \int_l^{H^*}$, then A/A^H is a separable extension. But more is true: in Theorem 1.15 we show that when k is a field under an additional assumption on H, for any Hopf subalgebra H' of H, $A^{H'}/A^H$ is a separable extension as well. In particular, if k is a field, H and H^* are both semisimple, and A/A^H is H^*-Galois, then the conclusion of 1.15 holds.

Since $A \#_\sigma H$ is an H-Galois extension, in Theorem 1.11 we use the main theorem to show that if the right integral of H is cocommutative and A has a central element of trace 1, then $A \#_\sigma H/A$ is a separable extension. Consequently, using [RR], we show in Theorem 1.23 that under the above hypotheses, various properties go up from A to $A \#_\sigma H$, thus generalizing results of [RR, So]; for example, we show that $\mathrm{gl.dim}\, A = \mathrm{gl.dim}\, A \#_\sigma H$.

0. Preliminaries

In this section we summarize some characteristics of Frobenius algebras, Galois extensions, semisimple extensions, and separable extensions.

Throughout, k is a commutative ring, H is a Hopf algebra over k, with comultiplication Δ, counit ε, and antipode S. We use the "sigma" notation of [Sw]; any tensor product not otherwise specified will be over k. $\int_l$ ($\int_r$) will denote the space of left (right) integrals in H; when we need to specify the Hopf algebra, we write $\int_l^H$ ($\int_r^H$). Also, A will denote a left H-module algebra.

Recall the definition of a Frobenius algebra:

Definition 0.1 [Par]. An algebra H is called *Frobenius* if it is finitely generated projective, and there is a right (or left) H-module isomorphism

$$\theta: H_H \to H_H^*$$

called the *Frobenius isomorphism*. The right H-module structure on H^* is given by $(b^* \leftarrow a)((c) = b^*(ac)$. Note that when H^* is a Hopf algebra, this action coincides with the usual one:

$$b^* \leftarrow a = \sum b^*_{(2)} \langle b^*_{(1)}, a \rangle.$$

0.2. We now summarize the facts we will be using for a Hopf algebra H which is a Frobenius algebra. For proofs of these see [Par]. First, note that any finite dimensional Hopf algebra over a field is Frobenius. Now, when H is Frobenius, so is H^*, and the Frobenius isomorphism for H is given by

$$\theta: H \to H^*, \qquad \theta(h) = h \rightharpoonup T$$

(a left H-module map in this case), where $T \in \int_l^{H^*}$ generates the space of left integrals in H^* over k. Also, there is a left integral $t \in \int_l$, which generates (over k) the space of left integrals in H, and for which

$$t \rightharpoonup T = \varepsilon.$$

Any finitely generated projective Hopf algebra over a commutative ring has a bijective antipode, and it is easy to check that in the above circumstances, the following connections hold:

$$T \rightharpoonup t = T \rightharpoonup S^{-1}(t) = 1 \tag{0.3}$$

and

$$T \leftharpoonup S^{-1}(t) = \varepsilon.$$

In view of this, we set $u \in \int_r$, $u = S^{-1}(t)$. This right integral will be used in the next section.

Whenever H has a left integral t as above, there is an element $\lambda \in H^*$ defined by $th = \lambda(h)t \ \forall h \in H$; λ is grouplike by its definition, and $\lambda(1)t = t$ implies $\lambda(1) = 1$, since $\int_l$ is free over k, spanned by t. This element is of crucial importance in defining the Morita context, for we use it to define an automorphism on H,

$$h^\lambda = \lambda \rightharpoonup h = \sum h_{(1)} \langle \lambda, h_{(2)} \rangle, \tag{0.4}$$

and this we extend to an automorphism of $A \# H$ via $(a \# h)^\lambda = a \# h^\lambda$ [CFM]. H is *unimodular* (i.e., $\int_l = \int_r$) iff $\lambda = \varepsilon$.

Note that for t, T as above, $T \leftharpoonup t = \lambda$, for if $g \in H$ then

$$\langle T \leftharpoonup t, g \rangle = \langle T, tg \rangle = \langle \lambda, g \rangle \langle T, t \rangle = \langle \lambda, g \rangle.$$

Here we have the easily checked formulae $\forall g \in H$:

(0.5)
(a) $S(g) = (T \leftharpoonup g) \rightharpoonup t$
(b) $S^{-1}(g^\lambda) = (g \rightharpoonup T) \rightharpoonup t$

and if T is also a right integral in H^*, then

(c) $S(g^\lambda) = t \leftharpoonup (g \rightharpoonup T)$
(d) A is an H^*-comodule algebra via

$$\rho(a) = \sum t_{(1)} \cdot a \otimes S^{-1}(t_{(2)} \rightharpoonup T) \qquad \forall a \in A.$$

Note that (a) implies that $S(t)=t^{\lambda}$. Let us write ah for $a \# h$, and denote by S the antipode of H^* as well as that of H (it will be clear by the context which is meant).

Recall that A is a left $A \# H$-module via the action

$$(ah)\cdot b=a(h\cdot b) \tag{0.6}$$

and a right $A \# H$-module via

$$a\cdot(bh)=S^{-1}(h)\cdot(ab). \tag{0.7}$$

In order to form the Morita context, we have defined a new right action of $A \# H$ on A [CFM]:

$$a\leftarrow(bh)=a\cdot(bh)^{\lambda}=S^{-1}(h^{\lambda})\cdot(ab). \tag{0.8}$$

The following context was first introduced in [CFM] for H finite dimensional over a field, and generalizes [CF].

THEOREM 0.9. *Let H be a Frobenius Hopf algebra, and A a left H-module algebra; let $t\in\int_l$ as in* (0.2). *Consider A as a left* (*right*) *A^H-module via left* (*right*) *multiplication, as a left $A \# H$-module as in* (0.6), *and a right $A \# H$-module as in* (0.8). *Then $[A^H, {}_{A^H}A_{A \# H}, {}_{A \# H}A_{A^H}, A \# H]$, together with the maps*

$$[\ ,\ ]: A\otimes_{A^H}A\to A \# H,\qquad [a,b]=atb \tag{0.10}$$

$$(\ ,\ ): A\otimes_{A \# H}A\to A^H,\qquad (a,b)=t\cdot(ab) \tag{0.11}$$

is a Morita context.

The proof in [CFM] carries through as is.

All of the results of [CFM, Sec. 2] now follow for the case of H Frobenius over a commutative ring k.

We now turn to Galois extensions.

DEFINITION 0.12 [KT]. Let H be a Hopf algebra, and A a right H-comodule algebra. $A/A^{\mathrm{co}\,H}$ is said to be *H-Galois* if the map $\beta: A\otimes_{A^{\mathrm{co}\,H}}A\to A\otimes H$, $\beta(a\otimes b)=\sum ab_{(0)}\otimes b_{(1)}$, is a bijection. Recall that $A^{\mathrm{co}\,H}=\{a\in A\mid \rho(a)=a\otimes 1\}$, and if H is finitely generated projective as an algebra, then $A^{\mathrm{co}\,H}=A^{H^*}$.

Remark 0.13. (a) A/A^H is H^*-Galois iff the map $[\ ,\ ]$ of the above Morita context is surjective [CFM, Th. 1.2]. So, let $\{x_i, y_i\}\subset A$ s.t. $\sum_i[x_i,y_i]=1$. Then $\forall a\in A$:

$$a=a\sum_i[x_i,y_i]=\sum_i(a,x_i)\,y_i=\sum_i t\cdot(ax_i)\,y_i$$

and

$$a=\sum_i [x_i, y_i]a=\sum_i x_i(y_i, a)=\sum_i x_i t\cdot(y_i a).$$

In particular, if $a=1$, $1=\sum(t\cdot x_i)\,y_i=\sum x_i(t\cdot y_i)$.

(b) Furthermore, surjectivity of $[\ ,\ ]$ implies its injectivity by the Morita theorems [Jac, Morita I]. Thus if $\{u_i, v_i\}\subset A$ are elements such that $\sum_i [u_i, v_i]\in C_{A\,\#\,H}(A)$, then

$$\forall a\in A\ :\ \sum au_i\otimes v_i=\sum u_i\otimes v_i a.$$

In particular, if $\sum[x_i, y_i]=1$, then $\sum ax_i\otimes y_i=\sum x_i\otimes y_i a$. Obviously, if A/A^H is H^*-Galois, and $\sum a_i\otimes b_i$ is a separability idempotent (as in the following definition), then $\omega=\sum[a_i, b_i]\in C_{A\,\#\,H}(A)$.

(c) If also the map $(\ ,\ )$ of the Morita context is onto, then the Morita theorems yield a bijective correspondence between H-stable right ideals of A, and H^*-stable right ideals of $A\ \#\ H$; also between the right ideals of A^H and the H-stable right ideals of A. Of course the left-hand version holds also.

So-called "Maschke type" theorems [Pas, CF, D] are actually instances of semisimple extensions.

DEFINITION 0.14 [HS]. Let A be a ring, and $B<A$ a subring. The extension A/B is said to be a *left semisimple extension* if every submodule of a left A-module which is a B-direct summand is an A-direct summand. (Right semisimple extensions are defined analogously.)

A *separable extension* of A over a (not necessarily commutative) ring B is defined as for separable extension over commutative rings [HS]—i.e., via the existence of a *separability idempotent*: $\sum_i a_i\otimes b_i\colon\in A\otimes_B A$ s.t. $\forall x\in A$

$$\sum xa_i\otimes b_i=\sum a_i\otimes b_i x \qquad \text{and} \qquad \sum a_i b_i=1. \tag{0.15}$$

Note that a separable extension is a left and right semisimple extension by [HS, 2.6]. The following is well known for B commutative, and holds for B noncommutative as well.

Remark 0.16. A/B is a separable extension iff the multiplication map $A\otimes_B A\xrightarrow{\mu} A\to 0$ splits as an A-bimodule map. The splitting map is then given by $p\colon A\to A\otimes_B A$, $p(a)=\sum a_i\otimes b_i a$.

One more structure we will consider is the generalized crossed product $A\ \#_\sigma H$, for H acting weakly on A, and for a normal cocycle σ (see

[BCM] for properties and discussion of $A \#_\sigma H$). We will assume throughout that the cocycles are normal and (convolution) invertible. The following technical lemma will be necessary:

LEMMA 0.17. *Assume H acts weakly on the algebra A and σ is an invertible cocycle. Then $\forall g, h, l, m \in H$:*

(1) $1 \#_\sigma gh = \sum (\sigma^{-1}(g_{(1)}, h_{(1)}) \#_\sigma g_{(2)})((1 \#_\sigma h_{(2)})$.

(2) $\sum \sigma^{-1}(h_{(1)}, l_{(1)})(h_{(2)} \cdot \sigma(l_{(2)}, m))$

$$= \sum \sigma(h_{(1)} l_{(1)}, m_{(1)})\, \sigma^{-1}(h_{(2)}, l_{(2)} m_{(2)}).$$

(3) $\sum \sigma^{-1}(h_{(1)}, l_{(1)} m_{(1)})(h_{(2)} \cdot \sigma^{-1}(l_{(2)}, m_{(2)}))$

$$= \sum \sigma^{-1}(h_{(1)} l_{(1)}, m)\, \sigma^{-1}(h_{(2)}, l_{(2)}).$$

Proof. (1) By direct computation

$$\sum (\sigma^{-1}(g_{(1)}, h_{(1)}) \#_\sigma g_{(2)})(1 \# h_{(2)})$$
$$= \sum \sigma^{-1}(g_{(1)}, h_{(1)})\, \sigma(g_{(2)}, h_{(2)}) \#_\sigma g_{(3)} h_{(3)} = 1 \#_\sigma gh.$$

(2) $\sum \sigma^{-1}(h_{(1)}, l_{(1)})(h_{(2)} \cdot \sigma(l_{(2)}, m))$ (since $\sigma * \sigma^{-1} = id$)

$$= \sum \sigma^{-1}(h_{(1)}, l_{(1)})(h_{(2)} \cdot \sigma(l_{(2)}, m_{(1)}))\, \sigma(h_{(3)}, l_{(3)} m_{(2)})\, \sigma^{-1}(h_{(4)}, l_{(4)} m_{(3)})$$

by the cocycle condition

$$= \sum \sigma^{-1}(h_{(1)}, l_{(1)})\, \sigma(h_{(2)}, l_{(2)})\, \sigma(h_{(3)} l_{(3)}, m_{(1)})\, \sigma^{-1}(h_{(4)}, l_{(4)} m_{(2)})$$
$$= \sum \sigma(h_{(1)} l_{(1)}, m_{(1)})\, \sigma^{-1}(h_{(2)}, l_{(2)} m_{(2)}).$$

(3) By [BM, 1.16]:

$$h \cdot \sigma^{-1}(l, m) = \sum \sigma(h_{(1)}, l_{(1)} m_{(1)})\, \sigma^{-1}(h_{(2)} l_{(2)}, m_{(2)})\, \sigma^{-1}(h_{(3)}, l_{(3)}).$$

So:

$$\sum \sigma^{-1}(h_{(1)}, l_{(1)} m_{(1)})(h_{(2)} \cdot \sigma^{-1}(l_{(2)}, m_{(2)}))$$
$$= \sum \sigma^{-1}(h_{(1)}, l_{(1)} m_{(1)})\, \sigma(h_{(2)}, l_{(2)} m_{(2)})\, \sigma^{-1}(h_{(3)} l_{(3)}, m_{(3)})\, \sigma^{-1}(h_{(4)}, l_{(4)})$$
$$= \sum \sigma^{-1}(h_{(1)}, l_{(1)} m)\, \sigma^{-1}(h_{(2)}, l_{(2)}). \quad \blacksquare$$

$A \#_\sigma H$ is a left H^*-module algebra via the action $h^* \rightharpoonup (a \#_\sigma g) = a \#_\sigma h^* \rightharpoonup g = \sum a \#_\sigma g_{(1)} \langle h^*, g_{(2)} \rangle$. Note that $h^* \rightharpoonup (a \#_\sigma g)(b \#_\sigma 1) = (a \#_\sigma (h^* \rightharpoonup g))(b \#_\sigma 1)$.

1. Semisimple Extensions

Throughout this section, k will be a commutative ring, and H a Hopf algebra, Frobenius as an algebra.

Let A be a left H-module algebra, and define the *trace map*, $\mathrm{tr}: A \to A^H$ via

$$\mathrm{tr}(a) = t \cdot a, \qquad a \in A. \tag{1.1}$$

Note that the existence of an element, $c \in A$ of trace 1, is equivalent to the map $(\, , \,): A \otimes_{A^H} A \to A^H$ being surjective.

Our main theorems in this section will generalize some results from H finite dimensional semisimple to the case when H is Frobenius and the map $(\, , \,)$ is onto (e.g., [CF, Th. 4; D2, Th. 4]). That this is indeed a generalization we see in the following example:

Example 1.2. Let k and K be fields, char $k = p$, and K a field extension of k. Let $\phi \in \mathrm{Aut}\, K$ be an automorphism that satisfies $\phi^p = 1$. Then $\{1, \phi, ..., \phi^{p-1}\}$ are linearly independent; set $G = \langle \phi \rangle$. Now consider $K \# kG$; $t = \sum \phi^i$ is cocommutative (since all of kG is), so take $a \in K$ with $t \cdot a = \sum \phi^i(a) \neq 0$, and set $c = (1/\sum \phi^i(a))a$. Then $t \cdot c = 1$, $H = kG$ is not semsimple; however, $K \# kG$ is semisimple by Theorem 1.11. ∎

Remark 1.3. When H is finite dimensional over a field k, it is well known that semisimplicity of H is equivalent to $\varepsilon(t) = 1$. When H is Frobenius and k is a commutative ring, if $\varepsilon(t) = 1$, H will be a separable extension of k via the separability idempotent $\sum t_{(1)} \otimes S(t_{(2)})$, and so a semisimple extension.

When $(\, , \,): A \otimes_{A^H} A \to A^H$ is surjective, we have the following result, which in the case of k a field is due to [D1].

Proposition 1.4. *Assume $(\, , \,)$ is surjective and V is a left A^H-module. Consider $A \otimes_{A^H} V$ as a left H module via H acting on A on the left. Then $V \cong (A \otimes_{A^H} V)^H$ via $\varphi: 1 \otimes V \to (A \otimes_{A^H} V)^H$, $\varphi(1 \otimes v) = 1 \otimes v$.*

Proof. Let $c \in A$ be a trace 1 element. Define $\psi: (A \otimes_{A^H} V)^H \to 1 \otimes_{A^H} V$ via

$$\psi \sum a_i \otimes v_i = tc \cdot \sum a_i \otimes v_i.$$

Then $\psi(\sum a_i \otimes v_i) = \sum t \cdot (ca_i) \otimes v_i = 1 \otimes \sum (t \cdot ca_i) \cdot v_i \in 1 \otimes V$ and ψ is the inverse to φ, for

$$\psi\varphi(1 \otimes v) = (t \cdot c) \otimes v = 1 \otimes v$$

and

$$\varphi\psi\left(\sum a_i \otimes v_i\right) = \varphi\left(tc \cdot \left(\sum a_i \otimes v_i\right)\right)$$

$$= \varphi\left(\sum (t_{(1)} \cdot c)\, t_{(2)} \cdot \left(\sum a_i \otimes v_i\right)\right)$$

$$\left(\text{since} \sum a_i \otimes v_i \in (A \otimes_{A^H} V)^H\right)$$

$$= \varphi\left(\sum (t_{(1)} \cdot c)\, \varepsilon(t_{(2)}) \sum a_i \otimes v_i\right)$$

$$= (t \cdot c) \sum a_i \otimes v_i = \sum a_i \otimes v_i. \quad \blacksquare$$

Recall that a total integral [D1] is a morphism $\phi: H^* \to A$ which is a right H^*-comodule map satisfying $\phi(\varepsilon) = 1$. Our next proposition connects the concept of total integrals and the map (,):

Proposition 1.5. *Let A be a left H-module algebra. Then:*

(1) *(,) is surjective $\Leftrightarrow$ there exists a total integral $\phi: H^* \to A$.*

(2) *There exists $c \in C_A(A^H)$ (= the centralizer of A^H in A) so that $t \cdot c = 1 \Leftrightarrow$ there exists a total integral $\phi: H^* \to C_A(A^H)$.*

Proof. (1) (,) is surjective $\Leftrightarrow$ there exists $c \in A$ s.t. $t \cdot c = 1$. Assuming there exists such an element, define $\phi: H^* \to A$ as follows: Set $\phi(T) = c$, and for $h^* \in H^*$ let $\phi(h^*) = \theta^{-1}(h^*) \cdot c$ (θ as in 0.2). Then ϕ is a left H-module map (since θ is)—hence a right H^*-comodule map. Also, $\phi(\varepsilon) = \theta^{-1}(\varepsilon) \cdot c = 1$, so ϕ is a total integral. Conversely, if $\phi: H^* \to A$ is a total integral, set $c = \phi(T)$, so that $t \cdot c = t \cdot \phi(T) = \phi(t \rightharpoonup T) = \phi(\varepsilon) = 1$.

(2) If $c \in C_A(A^H)$, then $\forall a \in A^H$, $h^* \in H^*$, and $h = \theta^{-1}(h^*)$:

$$a\phi(h^*) = a(h \cdot c) = h \cdot (ac) = h \cdot (ca) = (h \cdot c)a = \phi(h^*)a.$$

On the other hand, if $\phi(H^*) \subset C_A(A^H)$, then $c = \phi(T) \in C_A(A^H)$. $\blacksquare$

Assuming A/A^H to be a Galois extension, existence of trace 1 elements is equivalent to direct summand requirements on A^H:

Proposition 1.6. *Let A/A^H be H^*-Galois. Then:*

(1) *A^H is a right (left) A^H-direct summand of $A \Leftrightarrow A$ has a trace 1 element, c [KT].*

(2) *A^H is an A^H-bimodule direct summand of $A \Leftrightarrow A$ has a trace 1 element $c \in C_A(A^H)$.*

Proof. (1) We reprove (1), since this proof gives the element explicitly. ($\Rightarrow$) By [CFM, Th. 1.2], A/A^H is H^*-Galois implies that $A \# H \cong^{\pi}$ $\mathrm{End}(A_{A^H})$, with π the ring isomorphism defined by the left $A \# H$ action on A. Assuming A^H is an A^H-direct summand of A, there exists a right A^H-projection $p: A \to A^H$. Let $e = \pi^{-1}(p)$; then $\forall a \in A$, $x \in A^H: e(ax) = (e \cdot a)x$, and $e \cdot a \in A^H$.

Let $x_1, ..., x_r$, $y_1, ..., y_r \in A$ s.t. $\sum [x_i, y_i] = 1$. Then $\sum x_i(t \cdot y_i) = 1$ (by 0.13). Hence $1 = e \cdot 1 = \sum e \cdot x_i(t \cdot y_i) = t \cdot (\sum (e \cdot x_i) y_i)$. Set $c = \sum (e \cdot x_i) y_i$ to get the desired element.

The proof of the left analogue is similar.

Note that $e = tc$, for $\forall a \in A$, $a = \sum x_i t \cdot (h_i a)$ (by 0.13); hence $e \cdot a = \sum (e \cdot x_i) t \cdot (y_i a) = t \cdot (\sum (e \cdot x_i) y_i a) = tc \cdot a$. But π is an isomorphism; hence $e = tc$.

($\Leftarrow$) If $c \in A$ has trace 1, set $e = tc$. Then $\pi(e)$ is a right A^H-projection, since $\forall x \in A^H: \pi(e)(x) = e \cdot x = t \cdot (cx) = (t \cdot c)x = x$.

(2) ($\Rightarrow$) Let p, c, e be as in (1) with p an A^H-bimodule morphism (or equivalently, p commutes with A^H). Since π is a ring homomorphism, this implies that $tc = e = \pi^{-1}(p)$ commutes with A^H. But t commutes with A^H, so c does too.

($\Leftarrow$) If c commutes with A^H, so does $e = tc$; hence $p = \pi(e)$ is an A^H-bimodule projection.

Part (2) may also be shown as a consequence of Proposition 1.5 and [D1]. ∎

In the proof of (1), H^*-Galois is not necessary to prove ($\Leftarrow$); analogously, we have:

PROPOSITION 1.7. *Let M be a left $A \# H$-module, and assume there exists $c \in C_A(A^H)$ of trace 1. Then M^H is an A^H-direct summand of M.*

Proof. Define $p: M \to M^H$ by $p(m) = tc \cdot m$. Then p is an A^H-projection, for if $x \in A^H$, then $p(x \cdot m) = tcx \cdot m = txc \cdot m = x(tc \cdot m) = x \cdot p(m)$; also if $m \in M^H$ then $p(m) = tc \cdot m = \sum (t_{(1)} \cdot c)(t_{(2)} \cdot m) = (t \cdot c)m = m$. ∎

We now come to the central theorem of this paper. Recall $T \in \int_l^{H^*}$ and $T \rightharpoonup t = 1$ (0.3).

THEOREM 1.8. *Let H be a Hopf algebra, Frobenius as an algebra, and let A/A^H be an H^*-Galois extension ($\sum [x_i, y_i] = 1$). Then conditions (1)–(6) are equivalent:*

(1) *A/A^H is a separable extension.*

(2) *$\exists \omega \in C_{A \# H}(A)$ with $T \rightharpoonup \omega = 1$.*

(3) *A is an A-bimodule direct summand of $A \# H$.*

(4) *There exists an element $c \in A$ such that $\sum x_i c y_i = 1$.*

(5) *There exists an element $c \in C_A(A^H)$ such that $\sum x_i c y_i = 1$.*

(6) [DT, Th. 3.14] *There exists a total integral $\phi: H \to C_A(A^H)$.*

If any of the above hold, then A/A^H is a left and right semisimple extension. Moreover, if A^H is semisimple Artinian, then so is A.

Proof. By Remark 0.13(b), $[\ ,\ ]$ is bijective and we use this here.

$(1) \Rightarrow (2)$ Let $\sum a_i \otimes b_i \in A \otimes_{A^H} A$ be the separability idempotent for A/A^H, and set $\omega = \sum_{i=1}^m [a_i, b_i]$. Then $\omega \in C_{A \# H}(A)$ (by 0.13(b)), and $T \to \omega = \sum_{i=1}^m a_i (T \to t) b_i = \sum_{i=1}^m a_i b_i = 1$.

$(2) \Rightarrow (1)$ Let $\sum_{i=1}^m a_i \otimes b_i = [\ ,\ ]^{-1} \omega$. Then by 0.13(b)

$$\forall a \in A, \qquad \sum_{i=1}^m a a_i \otimes b_i = \sum_{i=1}^m a_i \otimes b_i a$$

and

$$\sum_{i==1}^m a_i b_i = T \to \sum_{i=1}^m a_i t b_i = T \to \omega = 1.$$

Hence A/A^H is a separable extension.

$(2) \Leftrightarrow (3)$ follows from Proposition 1.6(2), by taking $B = A \# H$, acted upon by the Hopf algebra H^*, and for which $B^{H^*} = (A \# H)^{H^*} \cong A$ (as rings, [BCM]).

$(5) \Rightarrow (4)$ is obvious.

$(4) \Rightarrow (1)$ Set $\sum_i a_i \otimes b_i = \sum_i x_i c \otimes y_i$. Then $\sum_i a_i \otimes b_i$ is a separability idempotent, for $\sum a_i b_i = \sum x_i c y_i = 1$, and if $a \in A$, then $\sum a x_i c \otimes y_i = \sum x_i c \otimes y_i a$ (by 0.13(b)).

$(1) \Rightarrow (5)$ Let $\sum a_i \otimes b_i$ be the separability idempotent, and set $c = \sum a_i (t \cdot b_i)$. Then

$$\sum_{i,j} x_i a_j (t \cdot b_j) y_i = \sum_{i,j} a_j t \cdot (b_j x_i) y_i = (\text{by } 0.13(\text{a})) \sum_j a_j b_j = 1;$$

furthermore, $c \in C_A(A^H)$, for if $a \in A^H$ then

$$\sum a a_i (t \cdot b_i) = (\text{by } 0.13(\text{b})) \sum a_i (t \cdot b_i a) = \sum a_i (t \cdot b_i) a.$$

$(5) \Leftrightarrow (6)$ Recall the right Miyashita–Ulbrich action of H^* on $C_A(A^H)$: if $h^* \in H^*$, let $\beta^{-1}(1 \otimes h^*) = \sum u_i \otimes v_i \in A \otimes_{A^H} A$. Then $x \cdot h^* = \sum u_i x v_i$ $\forall x \in C_A(A^H)$. Similarly to 0.7, this translates to a left action of H^* on $C_A(A^H)$ via $h^* \cdot x = x \cdot S^{-1}(h^*)$. Now, $\beta(\sum x_i \otimes y_i) = 1 \otimes S^{-1}(T)$, for by

0.5(d), if $a \in A$, then $\rho(a) = \sum t_{(1)} \cdot a \otimes S^{-1}(t_{(2)} \rightarrow T)$, so $\beta(a \otimes b) = \sum a(t_{(1)} \cdot b) \otimes S^{-1}(t_{(2)} \rightarrow T)$ $\forall a, b \in A$. Hence $\sum_{i,(t)} x_i(t_{(1)} \cdot y_i) \# t_{(2)} = \sum [x_i, y_i] = 1$ implies that $\beta(\sum_i x_i \otimes y_i) = 1 \otimes S^{-1}(T)$. Thus $T \cdot x = x \cdot S^{-1}(T) = \sum x_i x y_i$ $\forall x \in C_A(A^H)$. Proposition 1.5(1) applied to the H^*-module algebra $C_A(A^H)$ now yields (5)$\Leftrightarrow$(6). Note that this proof takes a different approach to that of [DT].

If any of the conditions (1)–(6) hold, then in particular (1) does—hence A/A^H is a left and right semisimple extension [HS, 2.6]. Thus complete reducibility as an A^H-module implies complete reducibility as an A-module. ▌

As an immediate corollary, we get the following generalization of [CF, BM]:

COROLLARY 1.9. *Let H act weakly on A, let σ be a normal invertible cocycle, and suppose $\varepsilon(t) = 1$. Then $A \#_\sigma H$ is a separable extension of A. In particular* [BM], *$A \#_\sigma H$ is a semisimple extension of A.*

Proof. By [BM, 1.18], $A \#_\sigma H/A$ is H-Galois. Now take $\omega = 1 \# 1 \# \varepsilon$. Then $t \rightarrow \omega = t \rightarrow \varepsilon := \varepsilon(t) = 1$, and $\omega \in C_{A \#_\sigma H \# H^*}(A \#_\sigma H)$ trivially. ▌

When H is not semisimple, but A has a trace 1 element, we get still another corollary relating A to $A \#_\sigma H$, under certain cirsumstances. An important condition here is that H has a right integral $t \in \int_r$ which is cocommutative: $\Delta(t) = \sum_{(t)} t_{(1)} \otimes t_{(2)} = \sum_{(t)} t_{(2)} \otimes t_{(1)}$.

The following was shown [LR, Prop. 8] to hold for H finite dimensional over a field k, using bilinear forms. Here we give a direct proof, generalizing that result:

PROPOSITION 1.10. *Let H be a Frobenius Hopf algebra and k a commutative ring, and $t \in \int_l$, $T \in \int_l^{H^*}$ as in* (0.2). *Then t is cocommutative iff T is also a right integral and $S^2 = id$.*

Proof. We use the formula $(T \leftarrow g) \rightarrow t = S(g)$ (0.5a). Assume t is cocommutative. Then

$$
\begin{aligned}
S(g) = (T \leftarrow g) \rightarrow t &= \sum t_{(1)} \langle T \leftarrow g, t_{(2)} \rangle \qquad \text{(by cocommutativity of } t) \\
&= \sum t_{(2)} \langle T \leftarrow g, t_{(1)} \rangle = \sum t_{(2)} \langle T, g t_{(1)} \rangle \\
&= \sum S^{-1}(g_{(3)})\, g_{(2)} t_{(2)} \langle T, g_{(1)} t_{(1)} \rangle \\
&= S^{-1}(g) \sum t_{(2)} \langle T, t_{(1)} \rangle \qquad \text{(by cocommutativity of } t) \\
&= S^{-1}(g)(T \rightarrow t) = S^{-1}(g).
\end{aligned}
$$

So $S^2 = id$.

To show that T is a 2-sided integral, recall that analogously to (0.2), the map $\theta' : H^* \to H$, $\theta'(h^*) = h^* \rightharpoonup t$, is a bijection. So, $\forall h^* \in H^* : \theta'(Th^*) = Th^* \rightharpoonup t = \langle Th^*, t\rangle$ (since Th^* is also a left integral) $= \sum \langle T, t_{(1)}\rangle \langle h^*, t_{(2)}\rangle = \sum \langle h^*, t_{(1)}\rangle\langle T, t_{(2)}\rangle = \langle h^*T, t\rangle = \langle h^*, 1\rangle\langle T, t\rangle = \langle h^*, 1\rangle$. But so is $\theta'(h^*T) = \langle h^*, 1\rangle T \rightharpoonup t = \langle h^*, 1\rangle$. So $h^*T = Th^*$, and T is 2-sided.

For the converse assume T is 2-sided, $S = S^{-1}$, and use (0.5)(b) and (c); let $g^*, h^* \in H^*$. Since comultiplication in H is dual to multiplication in H^*, it suffices to show that

$$\langle g^*h^*, t\rangle = \langle h^*g^*, t\rangle.$$

Let $h = \theta^{-1}(h^*)$, i.e., $h^* = h \rightharpoonup T$. Then

$$\begin{aligned}\langle g^*h^*, t\rangle &= \langle g^*, h^* \rightharpoonup t\rangle = \langle g^*, (h \rightharpoonup T) \rightharpoonup t\rangle \qquad \text{(by 0.5(b))}\\ &= \langle g^*, S^{-1}(h^{\lambda})\rangle\end{aligned}$$

and

$$\begin{aligned}\langle h^*g^*, t\rangle &= \langle g^*, t \leftharpoonup h^*\rangle = \langle g^*, t \leftharpoonup (h \rightharpoonup T)\rangle \qquad \text{(by 0.5(c))}\\ &= \langle g^*, S(h^{\lambda})\rangle \qquad (\text{since } S = S^{-1})\\ &= \langle g^*, S^{-1}(h^{\lambda})\rangle\end{aligned}$$

so the result follows. ∎

THEOREM 1.11. *Let A be a left H-module algebra with a central trace 1 element c. If $t \in \int_l$ is cocommutative and σ is an invertible normal 2-cocycle, then $A \#_\sigma H$ is a separable extension of A.*

Proof. Set $u = S(t)$; then u is cocommutative. As in Corollary 1.9, $A \#_\sigma H/A$ is H-Galois. We define an element $\omega \in A \#_\sigma H \# H^*$, satisfying the requirements of Theorem 1.8(2). Let

$$\omega = \sum_{(u)} (\sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \# T)(c \#_\sigma u_{(4)} \# \varepsilon).$$

Then

$$t \rightharpoonup \omega = \sum (\sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \# t \rightharpoonup T)(c \#_\sigma u_{(4)} \# \varepsilon).$$

So we calculate:

$$\begin{aligned}&\sum (\sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}))(c \#_\sigma u_{(4)})\\ &= \sum (\sigma^{-1}(S(u_{(4)}), u_{(5)})(S(u_{(3)}) \cdot c)\, \sigma(S(u_{(2)}), u_{(6)})) \#_\sigma S(u_{(1)})\, u_{(7)}\\ &= \sum (\sigma^{-1}(S(u_{(4)}), u_{(5)})(S(u_{(3)})\, u_{(6)} S(u_{(7)}) \cdot c)\, \sigma(S(u_{(2)}), u_{(8)})) \#_\sigma S(u_{(1)})\, u_{(9)}\end{aligned}$$

(by the twisted module condition)

$$=\sum (\sigma^{-1}(S(u_{(4)}), u_{(5)})\, \sigma(S(u_{(3)}), u_{(6)})\, (S(u_{(2)})\, u_{(8)} S(u_{(7)}) \cdot c) \#_\sigma S(u_{(1)})\, u_{(9)}$$

(since $S^{-1} = S$)

$$=\sum (S(u_{(2)}) \cdot c) \#_\sigma S(u_{(1)})\, u_{(3)} \quad \text{(since } u \text{ is cocommutative)}$$

$$=\sum S(u_{(1)}) \cdot c \#_\sigma S(u_{(3)})\, u_{(2)} = \sum S(u_{(1)}) \cdot c \#_\sigma S^{-1}(u_{(3)})\, u_{(2)}$$

$$= S(u) \cdot c = t \cdot c = 1.$$

To show that $\omega \in C_{A \#_\sigma H \# H^*}(A \#_\sigma H)$, by the bijectivity of [,], it suffices to show that if $\tilde{\omega} = \sum \sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \otimes_A (c \#_\sigma u_{(4)})$, then $\forall z \in A \#_\sigma H$, $(z \otimes 1)\tilde{\omega} = \tilde{\omega}(1 \otimes z)$ $\forall a \in A$.

First let $z = a \#_\sigma 1$. Then

$$\sum \sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \otimes_A (c \#_\sigma u_{(4)})(a \#_\sigma 1)$$

$$=\sum \sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \otimes_A c(u_{(4)} \cdot a) \#_\sigma u_{(5)}$$

(since $c \in Z(A)$, and the tensor is over $A = (A \# H)^{H^*}$)

$$=\sum (\sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}))(u_{(4)} \cdot a \#_\sigma 1) \otimes_A c \#_\sigma u_{(5)}$$

$$=\sum \sigma^{-1}(S(u_{(3)}), u_{(4)})(S(u_{(2)})\, u_{(5)} \cdot a) \#_\sigma S(u_{(1)}) \otimes_A c \#_\sigma u_{(6)}$$

(by the twisted module condition)

$$=\sum (S(u_{(3)})\, u_{(4)} \cdot a)\, \sigma^{-1}(S(u_{(2)}), u_{(5)}) \#_\sigma S(u_{(1)}) \otimes_A c \#_\sigma u_{(6)}$$

$$=\sum a\sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \otimes_A c \#_\sigma u_{(4)}.$$

Next, let $z = 1 \#_\sigma h$. Then:

$$=\sum \sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \otimes_A (c \#_\sigma u_{(4)})(1 \#_\sigma h)$$

$$=\sum \sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}) \otimes_A c\sigma(u_{(4)}, h_{(1)}) \#_\sigma u_{(5)} h_{(2)}$$

(since $c \in Z(A)$)

$$=\sum (\sigma^{-1}(S(u_{(2)}), u_{(3)}) \#_\sigma S(u_{(1)}))(\sigma(u_{(4)}, h_{(1)} \#_\sigma 1) \otimes_A c \#_\sigma u_{(5)} h_{(2)}$$

$$=\sum \sigma^{-1}(S(u_{(3)}), u_{(4)})(S(u_{(2)}) \cdot \sigma(u_{(5)}, h_{(1)}) \#_\sigma S(u_{(1)})) \otimes_A c \#_\sigma u_{(6)} h_{(2)}$$

(by Lemma 0.17(2))

$$=\sum \sigma(S(u_{(3)})\, u_{(4)}, h_{(1)})\, \sigma^{-1}(S(u_{(2)}), u_{(5)} h_{(2)}) \#_\sigma S(u_{(1)}) \otimes_A c \#_\sigma u_{(6)} h_{(3)}$$

$$=\sum \sigma^{-1}(S(u_{(2)}), u_{(3)}h_{(1)}) \#_\sigma S(u_{(1)}) \otimes_A c \#_\sigma u_{(4)}h_{(2)}$$
(by expanding h)

$$=\sum \sigma^{-1}(h_{(1)}S(u_{(2)}h_{(4)}), u_{(3)}h_{(5)}) \#_\sigma h_{(2)}(S(u_{(1)})h_{(3)}) \otimes_A c \#_\sigma u_{(4)}h_{(6)}$$

$$=\sum \sigma^{-1}(h_{(1)}S(u_{(2)}), u_{(3)}) \#_\sigma h_{(2)}S(u_{(1)}) \otimes_A c \#_\sigma u_{(4)} \quad \text{(by 0.17(1))}$$

$$=\sum (\sigma^{-1}(h_{(1)}S(u_{(3)}), u_{(4)})\, \sigma^{-1}(h_{(2)}, S(u_{(2)}) \#_\sigma h_{(3)}) \times (1 \#_\sigma S(u_{(1)}))) \otimes_A c \#_\sigma u_{(4)} \quad \text{(by 0.17(3))}$$

$$=\sum (\sigma^{-1}(h_{(1)}, S(u_{(3)})\, u_{(4)})(h_{(2)} \cdot \sigma^{-1}(S(u_{(2)}), u_{(5)})) \#_\sigma h_{(3)}) \times (1 \#_\sigma S(u_{(1)})) \otimes_A c \#_\sigma u_{(4)}$$

$$=\sum (1 \#_\sigma h)(\sigma^{-1}(S(u_{(2)}, u_{(3)}) \#_\sigma S(u_{(1)}))) \otimes_A c \#_\sigma u_{(4)}.$$

So $A \#_\sigma H$ is a separable extension of A. ∎

As an immediate corollary we now have:

COROLLARY 1.12. *Let A and H satisfy the conditions of the theorem. If A is semisimple Artinian, so is $A \#_\sigma H$.*

Another corollary of the central theorem is Doi's result:

COROLLARY 1.13 [D2, Th. 4]. *Let A/A^H be an H^*-Galois extension, with $\varepsilon(T) = \langle T, 1 \rangle = 1$. Then A/A^H is a separable extension. Consequently, A/A^H is a semisimple extension, and if A^H is semisimple Artinian, so is A.*

Proof. Letting $\omega = 1 \# 1$, condition (2) of the theorem is satisfied. ∎

More generally, we will consider $A^{H'}$ for any Hopf subalgebra $H' < H$ (in the above corollary $H' = k$, so $A^{H'} = A$).

A classical result of Galois theory states that if F is a field extension of K, F/K a finite Galois extension with $G = \mathrm{Gal}(F/K)$ and $G' < G$ a subgroup, then $F^{G'}/K$ is a separable extension. This was generalized in [HS, Th. 3.4] to an arbitrary algebra A. We now generalize to certain finite dimensional Hopf algebras H over a field k.

Assume that k is a field, and H is finite dimensional in 1.14–1.16.

The following proposition is a consequence of [NZ]:

PROPOSITION 1.14. *Let $H' < H$ be a Hopf subalgebra, $t \in \int_l^H$, $t' \in \int_l^{H'}$ left integrals of H and H', resp. Then:*

(1) [LR, Prop. 2.4]. *There exists* $y \in A$ *such that* $t = t'y$.

(2) *If* H^* *is semisimple, then there exists* $T \in \int_l^{H^*}$ *so that* $T \rightharpoonup t' = 1$.

Proof of (2). Since k is a field, it suffices to show that $T \rightharpoonup t' \neq 0$. Assume $T \rightharpoonup t' = 0$. Let $\Delta(t') = \sum s_i \otimes t_i$, with $\{s_i\}$ and $\{t_i\}$ bases of H'. Then $0 = T \rightharpoonup t' = \sum s_i \langle T, t_i \rangle$, the independence of $\{s_i\}$ implies $\langle T, t_i \rangle = 0$ $\forall i$—and so $\langle T, H' \rangle = 0$. But this contradicts $\varepsilon(T) = \langle T, 1 \rangle \neq 0$.

Using some ideas from the proof of [HS, Prop. 2.3] we show:

THEOREM 1.15. *Let* H *be a finite dimensional Hopf algebra over a field* k, *let* A/A^H *be* H^**-Galois, with a trace* 1 *element in* A, *and assume* H^* *is semisimple. If* $H' < H$ *is a Hopf subalgebra, for which there exists* $z \in H$ *so that* $t = t'y = zt'$, $t' \in \int_l^{H'}$, *then* $A^{H'}/A^H$ *is a separable extension.*

Remark 1.16. Note that if unimodularity of H implies unimodularity of H' for any Hopf subalgebra $H' < H$, then there is indeed such a $z \in H$, for then $t = S(t) = S(y)\, S(t') = S(y)t'$. This is not always true; for instance the Drinfeld double $D(H)$ is always unimodular [R, Th. 4(a)]. However, we ask the following:

Question 1.17. When does unimodularity of H imply unimodularity of every Hopf subalgebra $H' < H$?

Proof of Theorem 1.15. Let $\sum_{i=1}^m x_i \otimes y_i$ be the element for which $\sum_{i=1}^m [x_i, y_i] = 1$, and let $c \in A$ be an element of trace 1. Note that $1 = t \cdot c = t'y \cdot c = t' \cdot (y \cdot c)$, and let $b = y \cdot c$. We claim that

$$\sum t' \cdot x_i \otimes_{A^H} t' \cdot (y_i b)$$

is a separability idempotent for $A^{H'}/A^H$. First, to show that $\sum (t' \cdot x_i) t' \cdot (y_i b) = 1$, note that $\sum (t' \cdot x_i)\, y_i = 1$. For

$$\begin{aligned} \sum (t' \cdot x_i)\, y_i &= T \rightharpoonup \sum (t' \cdot x_i)\, t y_i \\ &= T \rightharpoonup \sum t' x_i t y_i = T \rightharpoonup t' = 1 \qquad \text{by Prop. 1.14(2)} \end{aligned}$$

So,

$$\sum (t' \cdot x_i)(t' \cdot y_i b) = t' \cdot \left(\sum (t' \cdot x_i)\, y_i b \right) = t' \cdot b = 1.$$

Next, in order to prove that $\forall \omega \in A^{H'}$,

$$\sum \omega (t' \cdot x_i) \otimes_{A^H} t' \cdot (y_i b) = \sum t' \cdot x_i \otimes_{A^H} t' \cdot (y_i b)\omega$$

we will first show the following equalities:

$\forall \omega \in A^{H'}$:

(a) $\sum_i (t' \cdot x_i) t \cdot (y_i b\omega) = \omega$,

(b) $\sum_i t \cdot (\omega x_i) t' \cdot (y_i b) = \omega$,

(c) $\forall i, j$: $t \cdot (y_i bt' \cdot (\omega x_j)) = t \cdot (t' \cdot (y_i b\omega) x_j)$.

(a) $\sum (t' \cdot x_i) t \cdot (y_i b\omega) = \sum t' \cdot (x_i t \cdot (y_i b\omega)) =$ (by 0.13) $t' \cdot (b\omega) = (t' \cdot b)\omega = \omega$.

(b) is proven in the same manner.

(c) Here we need $t = zt'$:

$$\begin{aligned} t \cdot (y_i bt' \cdot (\omega x_j)) &= t \cdot (y_i b\omega(t' \cdot x_j)) \\ &= zt' \cdot (y_i b\omega(t' \cdot x_j)) = z \cdot (t' \cdot (y_i b\omega)(t' \cdot x_j)) \\ &= zt' \cdot (t' \cdot (y_i b\omega) x_j) = t \cdot (t' \cdot (y_i b\omega) x_j). \end{aligned}$$

Now, the separability condition:

$$\begin{aligned} &\sum_i \omega(t' \cdot x_i) \otimes_{A^H} t' \cdot (y_i b) \\ &\quad = \sum_i t' \cdot (\omega x_i) \otimes_{A^H} t' \cdot (y_i b) \qquad \text{(by (a))} \\ &\quad = \sum_{i,j} (t' \cdot x_j) t \cdot (y_j bt' \cdot (\omega x_i)) \otimes_{A^H} t' \cdot (y_i b) \\ &\quad = \sum_{i,j} (t' \cdot x_j) \otimes_{A^H} t \cdot (y_j bt' \cdot (\omega x_i)) t' \cdot (y_i b) \qquad \text{(by (c))} \\ &\quad = \sum_{i,j} (t' \cdot x_j) \otimes_{A^H} t \cdot (t' \cdot (y_j b\omega) x_i) t' \cdot (y_i b) \qquad \text{(by (b))} \\ &\quad = \sum_j (t' \cdot x_j) \otimes_{A^H} t' \cdot (y_j b\omega) \\ &\quad = \sum_j t' \cdot x_j \otimes_{A^H} t' \cdot (y_j b)\omega. \quad \blacksquare \end{aligned}$$

COROLLARY 1.18 [HS, Prop. 3.4]. *If $H = kG$, G finite, A/A^G is Galois, and A has an element of trace* 1, *then for any subgroup $G' < G$, $A^{G'}/A^G$ is a separable extension.*

Proof. $H^* = (kG)^*$ is always semisimple, and for any $G' < G$, kG' is unimodular. Hence the result follows by Remark 1.16. ▮

As another corollary, we get the following theorem:

THEOREM 1.19. *Let H be a finite dimensional semisimple, cosemisimple Hopf algebra over a field k, and let A/A^H be H^*-Galois. Then for any Hopf subalgebra $H' < H$, $A^{H'}/A^H$ is a separable extension.*

Proof. The element $c=1$ is a trace 1 element in A since H is semisimple, and $\varepsilon(t') \neq 0$ since $\varepsilon(t) = \varepsilon(t')\,\varepsilon(y) \neq 0$, so H' is semisimple. Hence H' is unimodular, and the result follows by Remark 1.16. ▮

Returning to the general case, we infer further properties of $A \#_\sigma H$ from those of A, resulting from Theorem 1.8. We follow [RR], and consider the following conditions:

(Ai) A^H is an A^H-bimodule direct summand of A (equivalently, A has a trace 1 element $c \in C_A(A^H)$, by Prop. 1.6).

(Aii) The multiplication map $A \otimes_{A^H} A \xrightarrow{\mu} A \to 0$ is a split epimorphism of A-bimodules (equivalently, A/A^H is a separable extension, by 0.16).

(B) Given the induction and restriction functors:

$$T = A \otimes_{A^H} : \operatorname{mod} A^H \to \operatorname{mod} A$$
$$R = \text{restriction} : \operatorname{mod} A \to \operatorname{mod} A^H,$$

(T, R) is an adjoint pair of functors.

The following are generalizations of [So, RR]:

THEOREM 1.20. *Let H be a Hopf algebra, Frobenius as an algebra over a commutative ring k. Let A be a left H-module algebra, and σ a normal invertible cocycle. Then conditions* (Ai) *and* (Aii) *are satisfied with respect to H^* acting on $A \#_\sigma H$ and under either of the following*:

(1) $\varepsilon(t) = 1$, $t \in \int_l$.

(2) *A has a central element of trace* 1 *and $\int_l$ is cocommutative.*

Proof. Assume (1) holds. Then (Ai) holds by Proposition 1.6 with $c=1$ and (Aii) holds by Corollary 1.9.

Assume (2) holds. Then (Ai) holds by Proposition 1.6, and (Aii) holds by Theorem 1.11. ▮

THEOREM 1.21 [D2, Th. 5]. *Let H be Frobenius, and let A/A^H be H^*-Galois. Then condition* (B) *holds for right A and A^H modules.*

A proof similar to Doi's shows that (B) holds on the left as well.

THEOREM 1.22. *Let H be a Hopf algebra, Frobenius as an algebra, and A a left H-module algebra such that A/A^H is an H^*-Galois extension. Then A is* 1*-Gorenstein iff A^H is.*

Proof. By Theorem 1.21 condition (B) is satisfied; hence the result follows from [RR, Th. 1.3]. ▌

As an immediate consequence of [RR, Th. 1.3], and the fact that $A \#_\sigma H$ is H-Galois for any normal invertible cocycle σ, we get the following theorem:

THEOREM 1.23. *Let H be a Hopf algebra, Frobenius as an algebra, A a left H-module algebra, and σ an invertible normal cocylce. If either*

(i) $\varepsilon(t) = 1$, *or*

(ii) *A has a central element of trace* 1 *and $\int_l$ is cocommutative, then*:

(1) *If A is Artinian, then $A \#_\sigma H$ is of finite representation type iff A is.*

(2) gl. dim $A \#_\sigma H =$ gl. dim A.

(3) *$A \#_\sigma H$ is selfinjective iff A is.*

(4) *$A \#_\sigma H$ is an Auslander algebra iff A is.*

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